

Dirac equation: the stationary and dynamical scattering problems

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Dedicated to Heinz Langer on the occasion of his eightieth birthday with admiration.

Abstract

We prove that for the radial Dirac equation with Coulomb-type potential the generalized dynamical scattering operator coincides with the corresponding generalized stationary scattering operator. This fact is a quantum mechanical analogue of ergodic results in the classical mechanics.

MSC(2010): Primary 34L25, Secondary 34L05, 34L40.

Keywords: *Generalized wave operator, generalized scattering operator, deviation factor, Coulomb potential, ergodic theorems.*

1 Introduction

In the present paper we consider radial Dirac systems with Coulomb-type potentials:

$$\left(\frac{d}{dr} + \frac{k}{r}\right) f - (\lambda + m - v(r))g = 0, \quad (1.1)$$

$$\left(\frac{d}{dr} - \frac{k}{r}\right) g + (\lambda - m - v(r))f = 0, \quad k = \bar{k} \neq 0, \quad m > 0. \quad (1.2)$$

We assume that the potential $v(r)$ has the form

$$v(r) = -\frac{A}{r} + q(r), \quad A = \bar{A} \neq 0, \quad |k| > |A|. \quad (1.3)$$

We use the notions of the generalized wave operators, deviation factors and the generalized (dynamical) scattering operators S_{dyn} (see [12]). In Section 3 we introduce the notions of the generalized stationary scattering operators S_{st} and the corresponding deviation factors. The main result of this paper is the following ergodic type equality (see Theorem 5.1):

$$S_{dyn} = S_{st}. \quad (1.4)$$

Equality (1.4) is new even in the case $A = 0$ (in (1.3)), which is treated separately in Section 6.

Remark 1.1 *The ergodic theorems in classical mechanics assert that, under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average. In quantum mechanics, relation (1.4) is an analogue of the formulated ergodic properties from classical mechanics.*

2 Radial Dirac system

In this section, we study the asymptotic behavior of the solutions of radial Dirac system (1.1)–(1.3). Introduce the following notations

$$\gamma = \sqrt{k^2 - A^2} > 0, \quad \varepsilon = \sqrt{\lambda^2 - m^2} > 0 \quad (|\lambda| > m), \quad (2.1)$$

where i is the imaginary unit. We deal with the two cases:

$$\lambda > m, \quad \sqrt{m + \lambda} > 0, \quad -i\sqrt{m - \lambda} > 0, \quad (2.2)$$

and

$$\lambda < -m, \quad \sqrt{m - \lambda} > 0, \quad -i\sqrt{m + \lambda} > 0. \quad (2.3)$$

Further, $m > 0$ is fixed and the formulas below are valid (if not stated otherwise) for both cases. We consider solutions of (1.1), (1.2) depending on r , k and λ or on r , k and ε . It is easy to see that in both cases (2.2) and (2.3) the variable λ is uniquely recovered from ε .

1. We begin with the case when

$$v(r) = -\frac{A}{r} \quad (A = \overline{A} \neq 0, \quad |k| > |A|), \quad \text{i.e.,} \quad q(r) \equiv 0, \quad (2.4)$$

The regular at the point $r = 0$ solution $F_0 = \begin{bmatrix} f_0 \\ g_0 \end{bmatrix}$ of system (1.1), (1.2), with v of the form (2.4), satisfies the condition

$$F_0 = \begin{bmatrix} f_0 \\ g_0 \end{bmatrix} \sim N r^\gamma \begin{bmatrix} 1 \\ b_0 \end{bmatrix}, \quad b_0 := (\gamma + k)/A, \quad r \rightarrow 0, \quad (2.5)$$

where N does not depend on r and $N \neq 0$. Further we assume, that $N = 1$.

The solution F_0 can be represented in the form (see [3, Section 36]):

$$f_0 = \sqrt{m + \lambda} e^{-i\varepsilon r} r^\gamma (Q_1 + Q_2), \quad g_0 = -\sqrt{m - \lambda} e^{-i\varepsilon r} r^\gamma (Q_1 - Q_2). \quad (2.6)$$

The functions Q_1 and Q_2 can be expressed with the help of the confluent hypergeometric functions $\Phi(a, c, x)$ (see [2]):

$$Q_1 = a_1 \Phi(\gamma - A\lambda/\varepsilon, 2\gamma + 1, 2i\varepsilon r), \quad (2.7)$$

$$Q_2 = a_2 \Phi(\gamma + 1 - A\lambda/\varepsilon, 2\gamma + 1, 2i\varepsilon r). \quad (2.8)$$

Using relation (2.5) and equalities $N = 1$, $\Phi(a, c, 0) = 1$ we have

$$(a_2 + a_1)\sqrt{m + \lambda} = \frac{A}{\gamma + k}(a_2 - a_1)\sqrt{m - \lambda} = 1. \quad (2.9)$$

System (1.1), (1.2) (where v is given by (2.4)) admits also a non-regular at $r = 0$ solution $G_0 = \begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}$ of the form [1, 2]:

$$\phi_0 = \sqrt{m + \lambda} e^{-i\varepsilon r} r^\gamma (P_1 + P_2), \quad \psi_0 = -\sqrt{m - \lambda} e^{-i\varepsilon r} r^\gamma (P_1 - P_2), \quad (2.10)$$

where

$$P_1 = b_1 \Psi(\gamma + Ai\lambda/\varepsilon, 2\gamma + 1, 2i\varepsilon r), \quad (2.11)$$

$$P_2 = b_2 \Psi(\gamma + 1 + Ai\lambda/\varepsilon, 2\gamma + 1, 2i\varepsilon r), \quad (2.12)$$

and $\Psi(a, c, x)$ is the confluent hypergeometric function of the second kind. When $r \rightarrow 0$, we have (see [3, Ch. 6]):

$$\phi_0 \sim \sqrt{m + \lambda} r^{-\gamma} (b_1 + b_2), \quad \psi_0 \sim -\sqrt{m - \lambda} r^{-\gamma} (b_1 - b_2), \quad (2.13)$$

where

$$(b_2 + b_1)\sqrt{m + \lambda} = \frac{A}{-\gamma + k}(b_2 - b_1)\sqrt{m - \lambda}. \quad (2.14)$$

Using formulas (2.6)–(2.8) and asymptotic behavior of the confluent hypergeometric function (see [3]) we obtain the relations

$$f_0 = 2\Re(a_1\sqrt{m+\lambda}e^{-i\epsilon r}r^{-iA\lambda/2\epsilon}C_0(-\epsilon)(1+M_f/r+O(r^{-2}))), \quad (2.15)$$

$$g_0 = 2\Re(a_2\sqrt{m-\lambda}e^{i\epsilon r}r^{iA\lambda/2\epsilon}C_0(\epsilon)(1+M_g/r+O(r^{-2}))), \quad (2.16)$$

where M_f and M_g do not depend on r , $r \rightarrow \infty$, and

$$C_0(\epsilon) = \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+iA\lambda/\epsilon)}(2i\epsilon)^{-\gamma+iA\lambda/(2\epsilon)}. \quad (2.17)$$

Taking into account (2.10)–(2.12) and asymptotic behavior of the confluent hypergeometric function of the second kind (see [3]), we obtain the relations:

$$\phi_0 = b_1\sqrt{m+\lambda}e^{-i\epsilon r}(2i\epsilon)^{-\gamma}(2i\epsilon r)^{-iA\lambda/\epsilon}(1+M_\phi/r+O(r^{-2})), \quad (2.18)$$

$$\psi_0 = -b_1\sqrt{m-\lambda}e^{-i\epsilon r}(2i\epsilon)^{-\gamma}(2i\epsilon r)^{-iA\lambda/\epsilon}(1+M_\psi/r+O(r^{-2})), \quad (2.19)$$

where $r \rightarrow \infty$ and M_ϕ , M_ψ do not depend on r .

2. Now, we consider the case $q(r) \neq 0$. That is, we consider system (1.1), (1.2), where the initial v of the form (2.4) is perturbed by q and has the form (1.3). We study this case using solutions $F_0 = \begin{bmatrix} f_0 \\ g_0 \end{bmatrix}$ and $G_0 = \begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}$ of system (1.1), (1.2), (2.4) and assume that

$$\int_0^\infty (1+r)|q(r)|dr < \infty, \quad q(r) = \overline{q(r)}. \quad (2.20)$$

Introduce the 2×2 matrices $D(r, k, \epsilon)$ and $H(r)$ by the relations

$$D(r, k, \epsilon) = \begin{bmatrix} f_0(r, k, \epsilon) & \phi_0(r, k, \epsilon) \\ g_0(r, k, \epsilon) & \psi_0(r, k, \epsilon) \end{bmatrix}, \quad H(r) = \begin{bmatrix} 0 & q(r) \\ -q(r) & 0 \end{bmatrix}. \quad (2.21)$$

It is easy to see, that the solution $F(r, k, \epsilon)$ of the integral equation

$$F(r, k, \epsilon) = F_0(r, k, \epsilon) - \int_0^r D(r, k, \epsilon)D(t, k, \epsilon)^{-1}H(t)F(t, k, \epsilon)dt \quad (2.22)$$

satisfies the system (1.1)–(1.3) where (2.20) holds.

Proposition 2.1 *The solution $F(r, k, \epsilon)$ of (2.22) has the asymptotics*

$$F(r, k, \epsilon) \sim r^\gamma \begin{bmatrix} 1 \\ b_0 \end{bmatrix}, \quad r \rightarrow 0. \quad (2.23)$$

Proof. In view of (1.1), (1.2), $\det D(r)$ does not depend on r . Hence, using (2.5) and (2.13) we obtain

$$\det D(r, k, \varepsilon) = M_d(k, \varepsilon) \neq 0. \quad (2.24)$$

Thus, we have

$$r^{-\gamma} D(r) D^{-1}(t) t^\gamma = O(1), \quad r \geq t, \quad r \rightarrow 0. \quad (2.25)$$

The proposition follows from (2.20), (2.22) and (2.25). ■

Consider the solution Φ_0 of (1.1), (1.2), (2.4) with the asymptotics

$$\Phi_0(r, k, \varepsilon) = e^{-i\varepsilon r} r^{iA\lambda/\varepsilon} \begin{bmatrix} \sqrt{m+\lambda} + o(1) \\ -\sqrt{m-\lambda} + o(1) \end{bmatrix}, \quad r \rightarrow \infty. \quad (2.26)$$

The solution $\Phi(r, k, \varepsilon)$ of the integral equation

$$\Phi(r, k, \varepsilon) = \Phi_0(r, k, \varepsilon) + \int_r^\infty D(r, k, \varepsilon) D(t, k, \varepsilon)^{-1} H(t) \Phi(t, k, \varepsilon) dt \quad (2.27)$$

satisfies the system (1.1)–(1.3).

Proposition 2.2 *The solution $\Phi(r, k, \varepsilon)$ of (2.27) has the asymptotics*

$$\Phi(r, k, \varepsilon) = e^{-i\varepsilon r} r^{iA\lambda/\varepsilon} \begin{bmatrix} \sqrt{m+\lambda} + o(1) \\ -\sqrt{m-\lambda} + o(1) \end{bmatrix}, \quad r \rightarrow \infty. \quad (2.28)$$

Proof. Taking into account (2.15), (2.16) and (2.18), (2.19), we derive the relation

$$\|D(r) D^{-1}(t)\| = O(1) \quad (r \leq t), \quad r \rightarrow \infty. \quad (2.29)$$

The proposition follows from (2.27) and (2.29). ■

Let us consider $\Phi(r, k, \varepsilon) = \begin{bmatrix} \phi(r, k, \varepsilon) \\ \psi(r, k, \varepsilon) \end{bmatrix}$ in greater detail. Using (2.18)–(2.20) and (2.27), we obtain the assertion below.

Corollary 2.3 *The entries of the solution $\Phi(r, k, \varepsilon)$ of (2.27) have the following asymptotics:*

$$\phi = b_1 \sqrt{m+\lambda} e^{-\varepsilon r} (2\varepsilon)^{-\gamma} (2\varepsilon r)^{A\lambda/\varepsilon} (1 + M_\phi/r + O(m(r))), \quad (2.30)$$

$$\psi = -b_1 \sqrt{m-\lambda} e^{-\varepsilon r} (2\varepsilon)^{-\gamma} (2\varepsilon r)^{A\lambda/\varepsilon} (1 + M_\psi/r + O(n(r))), \quad (2.31)$$

where $r \rightarrow \infty$, M_ϕ and M_ψ do not depend on r , and the inequality

$$\int_a^\infty (|m(r)| + |n(r)|) dr < \infty \quad (2.32)$$

is valid for some $a > 0$.

3 Generalized stationary scattering operators

In many important cases, the initial and final states of the system (i.e., the states when $t \rightarrow \pm \infty$) cannot be regarded as free. For these cases the generalized dynamical wave operators and generalized dynamical scattering operators are used effectively instead of the usual dynamical wave and scattering operators (see [7, 11, 12]). In the present section we consider the radial Dirac system (1.1), (1.2) and introduce the notions of generalized wave and scattering operators for the stationary case. In this way, we deal with the non-free states when $r \rightarrow \pm \infty$ (instead of the non-free states when $t \rightarrow \pm \infty$ for dynamical systems). Let the following condition

$$\int_a^\infty |q'(r)|dr + \int_a^\infty |q^2(r)|dr + \int_b^a |q(r)|dr < \infty, \quad 0 < b < a < \infty, \quad (3.1)$$

where $q'(r) := (\frac{d}{dr}q)(r)$, be fulfilled. System (1.1), (1.2) can be written in the matrix form

$$\frac{d}{dr}Z = \mathcal{A}(r)Z, \quad \mathcal{A}(r) = \begin{bmatrix} -k/r & m + \lambda - v(r) \\ m - \lambda + v(r) & k/r \end{bmatrix}, \quad (3.2)$$

where $Z(r, k, \lambda) \in \mathbb{C}^2$. According to [4, Ch.II, Theorem 8], system (3.2) has two linear independent solutions Z_1 and Z_2 such that:

$$Z_1(r, k, \lambda) \sim \exp\{-i\theta\}V_0(r, \lambda)^{-1}C_1(k, \lambda), \quad r \rightarrow \infty; \quad (3.3)$$

$$Z_2(r, k, \lambda) \sim \exp\{i\theta\}V_0(r, \lambda)C_2(k, \lambda), \quad r \rightarrow \infty, \quad (3.4)$$

where $\theta = \varepsilon r$, $C_1(k, \lambda)$ and $C_2(k, \lambda)$ are 2×1 vectors, $C_1(k, \lambda) = \overline{C_2(k, \lambda)}$, ε is introduced in (2.1), and

$$V_0(r, \lambda) = \exp\left\{i\frac{\lambda}{\varepsilon} \int_a^r v(u)du\right\}. \quad (3.5)$$

Recall that we consider the cases (2.2) and (2.3), and (if not stated otherwise) our formulas are valid for both cases. Relations (3.3) and (3.4) yield the next assertion.

Proposition 3.1 *Let condition (3.1) be fulfilled. Then, the regular at the point $r = 0$ solution Z_{reg} of system (1.1)–(1.3) has the following asymptotics at $r \rightarrow \infty$:*

$$Z_{reg} \sim \frac{1}{2i} \left(\exp\{i\theta\}V_0(r, \lambda)C_2(k, \lambda) - \exp\{-i\theta\}V_0(r, \lambda)^{-1}C_1(k, \lambda) \right). \quad (3.6)$$

We introduce the scattering matrix function via the entries of

$$C_1(k, \lambda) = \begin{bmatrix} c_{1,1}(k, \lambda) \\ c_{2,1}(k, \lambda) \end{bmatrix}.$$

Definition 3.2 *The matrix function*

$$S(k, \lambda) = \begin{bmatrix} s_{1,1}(k, \lambda) & 0 \\ 0 & s_{2,1}(k, \lambda) \end{bmatrix}, \quad (3.7)$$

where

$$s_{n,1}(k, \lambda) := c_{n,1}(k, \lambda) / \overline{c_{n,1}(k, \lambda)} \quad (n = 1, 2), \quad (3.8)$$

is called the generalized stationary scattering matrix function.

Definition 3.3 *The function $V_0(r, \lambda)$ (see (3.5)) is called the stationary deviation factor.*

It follows from (3.7), (3.8) and the equality $C_1(k, \lambda) = \overline{C_2(k, \lambda)}$ that

$$S(k, \lambda)C_2(k, \lambda) = C_1(k, \lambda). \quad (3.9)$$

Remark 3.4 *Note that the deviation factor $V_0(r, \lambda)$ does not depend on k .*

4 Coulomb-type potentials: spectral theory

1. We study first the system (1.1), (1.2) where

$$k = 0, \quad v(r) \equiv 0, \quad (4.1)$$

that is, $k = 0$, $A = 0$, $q(r) \equiv 0$. In this case we have the system:

$$\frac{d}{dr}f - (\lambda + m)g = 0, \quad (4.2)$$

$$\frac{d}{dr}g + (\lambda - m)f = 0, \quad m > 0, \quad 0 \leq r < \infty. \quad (4.3)$$

Consider the special solution $f_1 = f$ and $g_1 = g$ of (4.2), (4.3) with the initial conditions

$$f_1(0, \lambda) = 1, \quad g_1(0, \lambda) = 0. \quad (4.4)$$

It follows from (4.2) and (4.3) that both $f(r, \lambda)$ and $g(r, \lambda)$ satisfy the equation:

$$\frac{d^2}{dr^2}y + (\lambda^2 - m^2)y = 0. \quad (4.5)$$

Taking into account (4.4) and (4.5), we obtain the equalities

$$f_1(r, \lambda) = \cos \varepsilon r, \quad g_1(r, \lambda) = \beta(\lambda) \sin \varepsilon r, \quad (4.6)$$

where ε is given in (2.1). Recall that either (2.2) or (2.3) holds. Relations (4.3) and (4.4) imply that

$$\left. \frac{d}{dr}g_1 \right|_{r=0} = m - \lambda. \quad (4.7)$$

It is immediate from (4.6) and (4.7) that

$$\beta(\lambda) = (m - \lambda)/\varepsilon. \quad (4.8)$$

2. Let us introduce the differential operator \mathcal{L}_0 , which corresponds to Dirac system (4.2), (4.3):

$$(\mathcal{L}_0 h)(r) = j_1 \frac{d}{dr}h(r) - m j_2 h(r) \quad (0 \leq r < \infty), \quad (4.9)$$

where

$$j_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad j_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad h(r) = \begin{bmatrix} h_1(r) \\ h_2(r) \end{bmatrix}. \quad (4.10)$$

The boundary condition is defined by the relation:

$$h_2(0) = 0. \quad (4.11)$$

By $\mu_1(u)$, we denote the spectral function of the operator \mathcal{L}_0 . Using (4.6) and (4.8), it is easy to see that the spectral density $\rho_1(\sigma) = \mu'_1(\sigma)$ is given by the formula

$$\rho_1(\sigma) = \frac{1}{\pi} \sqrt{\left| \frac{\sigma + m}{\sigma - m} \right|} \quad \text{for } |\sigma| > m; \quad \rho_1(\sigma) \equiv 0 \quad \text{for } |\sigma| < m. \quad (4.12)$$

The operator \mathcal{L}_0 is similar to the multiplication by λ in a space of functions $F(\lambda)$ ($\lambda \in E$) where

$$E := (-\infty, -m] \cup [m, +\infty). \quad (4.13)$$

More precisely, we have the following proposition.

Proposition 4.1 *The operator \mathcal{L}_0 , introduced by (4.9), (4.11), admits representation*

$$\mathcal{L}_0 = U_0 Q U_0^{-1}, \quad (4.14)$$

where the operators U_0 , U_0^{-1} and Q are given by the equalities:

$$(U_0 F)(r) = \int_E \begin{bmatrix} f_1(r, \lambda) \\ g_1(r, \lambda) \end{bmatrix} F(\lambda) \rho_1(\lambda) d\lambda = h(r), \quad (4.15)$$

$$(U_0^{-1} h)(\lambda) = \int_0^\infty \begin{bmatrix} f_1(r, \lambda) & g_1(r, \lambda) \end{bmatrix} h(r) dr = F(\lambda), \quad (4.16)$$

$$(QF)(\lambda) = \lambda F(\lambda) \quad (\lambda \in E). \quad (4.17)$$

Proof. In view of (4.9), (4.15) and (4.17), direct calculation shows that

$$\mathcal{L}_0 U_0 F = U_0 Q F. \quad (4.18)$$

■

We note that the following Parseval-type relation holds (see [14, Ch. 10]).

Proposition 4.2 *Let equality (4.15) be valid. Then, we have*

$$\int_E |F(\lambda)|^2 \rho_1(\lambda) d\lambda = \int_0^\infty (|h_1(r)|^2 + |h_2(r)|^2) dr, \quad (4.19)$$

where h_k ($k = 1, 2$) are the entries of h .

3. Next, we consider the radial Dirac system (1.1)–(1.3) assuming that (2.20) holds. We introduce the differential operator

$$\mathcal{L}h(r) = j_1 \frac{d}{dr} h(r) - m j_2 + V(r) h(r), \quad 0 \leq r < \infty, \quad (4.20)$$

where

$$V(r) = \begin{bmatrix} v(r) & k/r \\ k/r & v(r) \end{bmatrix}, \quad h(r) = \begin{bmatrix} h_1(r) \\ h_2(r) \end{bmatrix}. \quad (4.21)$$

Similar to (4.11), the boundary condition for \mathcal{L} is given by the relation:

$$h_2(0) = 0. \quad (4.22)$$

By G we denote the maximal invariant subspace on which the operator \mathcal{L} induces an operator with absolutely continuous spectrum, and P stands for the orthogonal projection from $L_2^2(0, \infty)$ onto G . The spectral function of the operator \mathcal{L} (given by (4.20) and (4.22)) is denoted by $\mu(\lambda)$ and the spectral density $\mu'(\lambda)$ (on the absolutely continuous part of spectrum) is denoted by $\rho(\lambda)$.

Proposition 4.3 *Let the operator \mathcal{L} be given by (4.20)–(4.22) and (1.3). Assume that (2.20) holds. Then, the equality*

$$\mathcal{L}P = UQU^{-1}P, \quad (4.23)$$

where the operators U , U^{-1} and Q have the form

$$(UF)(r) = \int_E \begin{bmatrix} f(r, \lambda) \\ g(r, \lambda) \end{bmatrix} F(\lambda) \rho(\lambda) d\lambda = h(r) \in G, \quad (4.24)$$

$$(U^{-1}h)(\lambda) = \int_0^\infty \begin{bmatrix} f(r, \lambda) & g(r, \lambda) \end{bmatrix} h(r) dr = F(\lambda) \quad (h \in G), \quad (4.25)$$

$$(QF)(\lambda) = \lambda F(\lambda), \quad \lambda \in E, \quad E := (-\infty, -m] \cup [m, +\infty), \quad (4.26)$$

is valid.

Proof. Similar to the proof of Proposition 4.1, direct calculation (using (4.20), (4.24) and (4.26) shows that

$$\mathcal{L}UF = UQF. \quad (4.27)$$

■

The following Parseval-type relation is fulfilled (see [14, Ch. 10]).

Proposition 4.4 *Let the conditions of Proposition 4.3 hold. Then,*

$$\int_E |F(\lambda)|^2 \rho(\lambda) d\lambda = \int_0^\infty (|h_1(r)|^2 + |h_2(r)|^2) dr. \quad (4.28)$$

Remark 4.5 *According to (4.14) and (4.23), we have*

$$e^{it\mathcal{L}_0} e^{-it\mathcal{L}} P = U_0 e^{itQ} U_0^{-1} U e^{-itQ} U^{-1} P. \quad (4.29)$$

4. Taking into account (2.28), we see that (under condition (2.17) instead of condition (3.1) in Section 3) the relations (3.6)–(3.9) are valid for V_0 of the form

$$V_0(r, \lambda) = r^{iA\lambda/\varepsilon}. \quad (4.30)$$

Using (3.6)–(3.9), we consider below the generalized stationary scattering matrix $S_{st}(\mathcal{L}, \mathcal{L}_0) = S(k, \lambda)$ for the case of Coulomb-type potentials satisfying (2.17).

In view of Corollary 2.3, we rewrite (3.6) as

$$Z_{reg} = \frac{1}{2i} \omega(k, \varepsilon) (\overline{\Phi(r, k, \varepsilon)} - \Phi(r, k, \varepsilon)), \quad (4.31)$$

where $\omega(k, \varepsilon)$ is a real-valued function. It follows from (2.30) and (2.31) that

$$c_{2,1}(k, \lambda) / c_{1,1}(k, \lambda) = i\beta(\lambda), \quad (4.32)$$

where β coincides with β in (4.8). Relations (3.8) and (4.32) imply that

$$s_{1,1}(k, \lambda) = -s_{2,1}(k, \lambda). \quad (4.33)$$

Hence, the scattering matrix $S(k, \lambda)$ ($\lambda = \bar{\lambda}$) has the form

$$S_{st}(\mathcal{L}, \mathcal{L}_0) := S(k, \lambda) = \begin{bmatrix} s_{1,1}(k, \lambda) & 0 \\ 0 & -s_{1,1}(k, \lambda) \end{bmatrix}, \quad |\lambda| > m. \quad (4.34)$$

5 Ergodic properties

In the present section, we consider the generalized dynamical scattering operators (which are introduced in Appendix) for the case of Dirac systems with Coulomb-type potentials, where $A_0 = \mathcal{L}_0 = \mathcal{L}_0^*$ and $A = \mathcal{L} = \mathcal{L}^*$, that is, we consider $S(\mathcal{L}, \mathcal{L}_0)$ given by (7.4) and (4.9), (4.20). Moreover, we consider $S(\mathcal{L}, \mathcal{L}_0)$ in momentum representation

$$S_{dyn}(\mathcal{L}, \mathcal{L}_0) = U_0^{-1} S(\mathcal{L}, \mathcal{L}_0) U_0, \quad (5.1)$$

where U_0 and U_0^{-1} are given by (4.15) and (4.16). We compare the generalized dynamical scattering operator S_{dyn} with the generalized stationary scattering operator S_{st} given by (4.34). More precisely, we compare the actions of S_{dyn}

and S_{st} on the subspace L of functions $f(\lambda) \in \mathbb{C}^2$ ($\lambda \in \tilde{E}$), where $\tilde{E} = (-\infty, -m) \cup (m, \infty)$:

$$L = \left\{ f(\lambda) = \begin{bmatrix} f_1(\lambda) \\ f_2(\lambda) \end{bmatrix} : f_1(\lambda) \equiv 0 \text{ for } \lambda < -m, \ f_2(\lambda) \equiv 0 \text{ for } \lambda > m \right\}. \quad (5.2)$$

In this way, we find formulas, which demonstrate quantum analogues of the ergodic properties from classical mechanics.

Theorem 5.1 *Let the radial Dirac system (1.1)-(1.3) and corresponding operators \mathcal{L}_0 and \mathcal{L} (defined via (4.9), (4.11) and (4.20), (4.22), respectively) be given. Assume that (2.20) holds. Then, the generalized stationary and dynamical scattering matrices are equal on L , that is,*

$$S_{st}(\mathcal{L}, \mathcal{L}_0)f = S_{dyn}(\mathcal{L}, \mathcal{L}_0)f \quad \text{for } f \in L. \quad (5.3)$$

Proof. Step 1. First, we study the operator $T = U_0^{-1}U$, where U_0^{-1} and U are given by (4.16) and (4.24). According to (4.16) and (4.24), T admits the representation

$$(TF)(\lambda) = \int_E F(u) \rho(u) \int_0^\infty (f_1(r, \lambda) f(r, u) + g_1(r, \lambda) g(r, u)) dr du. \quad (5.4)$$

Using (4.6) and (5.4), we rewrite the operator T in the form

$$T = T_1 + T_2, \quad (5.5)$$

where the operators T_1 and T_2 are defined by the formulas

$$(T_1 F)(\lambda) = \frac{d}{d\varepsilon} \int_E F(u) T_1(\varepsilon, u) du, \quad (5.6)$$

$$T_1(\varepsilon, u) := \rho(u) \int_0^\infty f(r, u) \frac{\sin(\varepsilon r)}{r} dr; \quad (5.7)$$

$$(T_2 F)(\lambda) = \beta(\lambda) \frac{d}{d\varepsilon} \int_E F(u) T_2(\varepsilon, u) du, \quad (5.8)$$

$$T_2(\varepsilon, u) := \rho(u) \int_0^\infty g(r, u) \frac{1 - \cos(\varepsilon r)}{r} dr. \quad (5.9)$$

We note that $\beta(\lambda)$ in (5.8) is given by (4.8).

We shall need some properties of the operators

$$R_l g = \frac{d}{d\varepsilon} \int_m^\infty g(u) R_l(\varepsilon, u) du, \quad l = 0, 1, 2, \quad (5.10)$$

where the kernels $R_l(\varepsilon, u)$ have the form

$$R_l(\varepsilon, u) = \rho(u) \int_0^\infty p_l(r, u) \frac{1 - \cos \varepsilon r}{r} dr. \quad (5.11)$$

Here, ρ is again the spectral density of \mathcal{L} , the functions p_0 and p_1 are fixed, and p_2 is some summable function:

$$p_0(r, u) = e^{-i\varepsilon r} r^{-iAu} / \sqrt{u^2 - m^2}, \quad (5.12)$$

$$p_1(r, u) = p_0(r, u)/r, \quad \int_0^\infty |p_2(r, \varepsilon)| dr < \infty. \quad (5.13)$$

We assume that the functions $g(u)$ belong to the class S , that is, $g \in C^\infty$ and functions g have finite support (more precisely, $g(u) \equiv 0$ for $u \notin (a_g, b_g)$, where $m < a_g < b_g < \infty$). It follows from (5.10)–(5.13) that

$$R_l(e^{it\sqrt{u^2 - m^2}} g(u)) \rightarrow 0, \quad t \rightarrow \pm \infty, \quad l = 1, 2. \quad (5.14)$$

Now, consider R_0 . Recall the relation [8, Ch. 2]:

$$\int_0^\infty r^\mu e^{ixr} dr = i\Gamma(\mu + 1) (e^{i\mu\pi/2} x_+^{-\mu-1} - e^{-i\mu\pi/2} x_-^{-\mu-1}), \quad (5.15)$$

where $\mu \neq -1, -2, \dots$; $\Gamma(\zeta)$ is Euler gamma function; $x_+ = x$ if $x > 0$ and $x_+ = 0$ if $x < 0$, $x_- = 0$ if $x > 0$ and $x_- = |x|$ if $x < 0$. Due to relations (5.10)–(5.12) the operator R_0 can be written in the form

$$R_0 g = \frac{1}{2} \frac{d}{d\varepsilon} \int_m^\infty g(u) (\Phi(u, -\varepsilon - \sqrt{u^2 - m^2}) + \Phi(u, \varepsilon - \sqrt{u^2 - m^2})) du, \quad (5.16)$$

where

$$\Phi(u, \zeta) = -\Gamma(-i\phi(u)) \rho(u) (e^{\pi\phi(u)/2} \zeta_+^{i\phi(u)} + e^{-\pi\phi(u)/2} \zeta_-^{i\phi(u)}), \quad (5.17)$$

$$\phi(u) = \frac{Au}{\sqrt{u^2 - m^2}}. \quad (5.18)$$

Introduce the operators

$$R_{\pm}g = \frac{1}{2} \frac{d}{d\varepsilon} \int_m^\infty g(u) \Phi(u, -\sqrt{u^2 - m^2} \mp \varepsilon) du, \quad (5.19)$$

According to (5.19), we have

$$R_0 = R_- + R_+. \quad (5.20)$$

It is easy to see that

$$R_+(e^{it\sqrt{u^2 - m^2}} g(u)) \rightarrow 0, \quad t \rightarrow \pm \infty. \quad (5.21)$$

Step 2. In this and the following steps of proof, we consider the case (2.2), where $\lambda > m$. The case (2.3) may be considered in the same way. The present step of proof is dedicated to the study of R_- . Taking into account (5.17) and (5.19) we obtain

$$R_- = V_1 + V_2, \quad (5.22)$$

$$V_1 g = \frac{1}{2} \frac{d}{d\varepsilon} \int_m^{\sqrt{m^2 + \varepsilon^2}} g(u) \xi_1(u) (\varepsilon - \sqrt{u^2 - m^2})^{i\phi(u)} du, \quad (5.23)$$

$$V_2 g = \frac{1}{2} \frac{d}{d\varepsilon} \int_{\sqrt{m^2 + \varepsilon^2}}^\infty g(u) \xi_2(u) (\sqrt{u^2 - m^2} - \varepsilon)^{i\phi(u)} du, \quad (5.24)$$

where the functions $\xi_1(u)$ and $\xi_2(u)$ are defined by the relations

$$\xi_1(u) = -\Gamma(-i\phi(u)) \rho(u) e^{\pi\phi(u)/2}, \quad (5.25)$$

$$\xi_2(u) = -\Gamma(-i\phi(u)) \rho(u) e^{-\pi\phi(u)/2}. \quad (5.26)$$

The operator V_1 can be written in the form

$$\begin{aligned} V_1 g = \frac{1}{2} \frac{d}{d\varepsilon} \int_0^\varepsilon g(\sqrt{m^2 + \eta^2}) \xi_1(\sqrt{m^2 + \eta^2}) \\ \times (\varepsilon - \eta)^{i\phi(\sqrt{m^2 + \eta^2})} (\eta / \sqrt{m^2 + \eta^2}) d\eta. \end{aligned} \quad (5.27)$$

Using [11, f-las (3.14) and (3.15)], we obtain (for $t \rightarrow \pm \infty$):

$$V_1(e^{it\sqrt{u^2 - m^2}} g(u)) \sim \frac{\varepsilon}{2\lambda} e^{it\varepsilon} |t|^{-i\phi(\lambda)} \Gamma(1 + i\phi(\lambda)) e^{\pm\pi\phi(\lambda)/2} \xi_1(\lambda) g(\lambda), \quad (5.28)$$

where $\lambda > m$ and the functions $g(u)$ belong to the class S . Recalling the well-known relation

$$\Gamma(1 + i\phi(\lambda))\Gamma(-i\phi(\lambda)) = \frac{i\pi}{\sinh(\pi\phi(\lambda))}, \quad (5.29)$$

from (5.25) and (5.28) we derive (for $t \rightarrow \pm\infty$):

$$|t|^{i\phi(\lambda)} e^{-it\varepsilon} V_1(e^{it\sqrt{u^2-m^2}} g(u)) \sim \nu_1(\lambda) e^{\pm\pi\phi(\lambda)/2} g(\lambda), \quad (5.30)$$

where

$$\nu_1(\lambda) = -i\rho(\lambda) \frac{\pi\varepsilon}{2\lambda \sinh(\pi\phi(\lambda))} e^{\pi\phi(\lambda)/2}. \quad (5.31)$$

The operator V_2 given by (5.24) admits representation

$$\begin{aligned} V_2 g &= \frac{1}{2} \frac{d}{d\varepsilon} \int_{\varepsilon}^{\infty} g(\sqrt{\eta^2 + m^2}) \xi_2(\sqrt{\eta^2 + m^2}) \\ &\quad \times (\eta - \varepsilon)^{i\phi(\sqrt{\eta^2 + m^2})} (\eta / \sqrt{\eta^2 + m^2}) d\eta. \end{aligned} \quad (5.32)$$

Due to (5.32) we have

$$V_2(e^{it\sqrt{u^2-m^2}} g(u)) \sim \lim_{\delta \rightarrow +0} \frac{\varepsilon}{2i\lambda} \phi(\lambda) g(\lambda) \xi_2(\lambda) \int_{\varepsilon}^{\infty} e^{it\eta} (\eta - \varepsilon)^{i\phi(\lambda)+\delta-1} d\eta. \quad (5.33)$$

Using (5.15) and the equality

$$\int_{\varepsilon}^{\infty} e^{it\eta} (\eta - \varepsilon)^{i\phi(\lambda)+\delta-1} d\eta = e^{it\varepsilon} \int_0^{\infty} e^{itr} r^{i\phi(\lambda)+\delta-1} dr, \quad (5.34)$$

we obtain

$$\begin{aligned} \int_{\varepsilon}^{\infty} e^{it\eta} (\eta - \varepsilon)^{i\phi(\lambda)+\delta-1} d\eta &\rightarrow e^{it\varepsilon} \Gamma(i\phi(\lambda)) \\ &\quad \times (e^{-\pi\phi(\lambda)/2} t_+^{-i\phi(\lambda)} + e^{\pi\phi(\lambda)/2} t_-^{-i\phi(\lambda)}), \end{aligned} \quad (5.35)$$

where $\delta \rightarrow +0$. It follows from (5.33) and (5.35) that for $t \rightarrow \pm\infty$ we have:

$$V_2(e^{it\sqrt{u^2-m^2}} g(u)) \sim -\frac{\varepsilon}{2\lambda} e^{it\varepsilon} |t|^{-i\phi(\lambda)} g(\lambda) \xi_2(\lambda) \Gamma(1 + i\phi(\lambda)) e^{\mp\pi\phi(\lambda)/2}. \quad (5.36)$$

Relations (5.26) and (5.36) imply that

$$|t|^{i\phi(\lambda)} e^{-it\varepsilon} V_2(e^{it\sqrt{u^2-m^2}} g(u)) \sim \nu_2(\lambda) e^{\mp\pi\phi(\lambda)/2} g(\lambda) \quad (t \rightarrow \pm\infty), \quad (5.37)$$

where

$$\nu_2(\lambda) = i\rho(\lambda) \frac{\pi\varepsilon}{2\lambda \sinh(\pi\phi(\lambda))} e^{-\pi\phi(\lambda)/2}. \quad (5.38)$$

Finally, taking into account (5.20)–(5.22), (5.30) and (5.37) we have

$$R_0(e^{it\sqrt{u^2-m^2}}g) \sim \frac{\pi\varepsilon}{i\lambda} |t|^{-i\phi(\lambda)} e^{it\varepsilon} \rho(\lambda) g(\lambda), \quad t \rightarrow +\infty, \quad (5.39)$$

and

$$R_0(e^{it\sqrt{u^2-m^2}}g(u)) \sim 0, \quad t \rightarrow -\infty. \quad (5.40)$$

It is easy to see that the equality

$$\overline{R_0}(e^{it\sqrt{u^2-m^2}}g(u)) = \overline{R_0(e^{-it\sqrt{u^2-m^2}}\overline{g(u)})}$$

holds for the operator $\overline{R_0}$ given by

$$\overline{R_0}f = \frac{1}{2} \frac{d}{d\varepsilon} \int_0^\infty f(u) \overline{R_0(\varepsilon, u)} du. \quad (5.41)$$

Thus, it follows from (5.39) and (5.40) that

$$\overline{R_0}(e^{it\sqrt{u^2-m^2}}g(u)) \sim i \frac{\pi\varepsilon}{\lambda} |t|^{i\phi(\lambda)} e^{it\varepsilon} \rho(\lambda) g(\lambda), \quad t \rightarrow -\infty; \quad (5.42)$$

$$\overline{R_0}(e^{it\sqrt{u^2-m^2}}g(u)) \sim 0, \quad t \rightarrow +\infty. \quad (5.43)$$

Step 3. Now, we return to the study of T . Relations (3.6), (5.4)–(5.9) and (5.39)–(5.43) imply that

$$T(e^{it\sqrt{u^2-m^2}}g) \sim \frac{\pi\varepsilon}{2\lambda} |t|^{-i\phi(\lambda)} e^{it\varepsilon} \rho(\lambda) (c_{2,1}(k, \lambda)\beta(\lambda) + ic_{1,1}(k, \lambda))g(\lambda), \quad (5.44)$$

where $t \rightarrow +\infty$. According to (2.1), (4.8) and (4.32) the equality

$$c_{2,1}(k, \lambda)\beta(\lambda) + ic_{1,1}(k, \lambda) = i \frac{2\lambda}{\lambda + m} c_{1,1}(k, \lambda) \quad (5.45)$$

is valid. Hence, relation (5.44) takes the form

$$T(e^{it\sqrt{u^2-m^2}}g) \sim i |t|^{-i\phi(\lambda)} e^{it\varepsilon} (\rho(\lambda)/\rho_1(\lambda)) c_{1,1}(k, \lambda) g(\lambda), \quad (5.46)$$

where $t \rightarrow +\infty$ and the function $\rho_1(\lambda)$ is given by (4.12). It follows from (5.42) and (5.45) that

$$T(e^{it\sqrt{u^2-m^2}}g) \sim -i |t|^{i\phi(\lambda)} e^{it\varepsilon} (\rho(\lambda)/\rho_1(\lambda)) \overline{c_{1,1}(k, \lambda)} g(\lambda), \quad (5.47)$$

where $t \rightarrow -\infty$. Using (5.46), (5.47) and the invariance principle for generalized wave operators (see [13, Theorem 1.1]), we obtain the relations

$$T(e^{itu}g) \sim i|t\lambda/\varepsilon|^{-i\phi(\lambda)} e^{it\lambda} (\rho(\lambda)/\rho_1(\lambda)) c_{1,1}(k, \lambda) g(\lambda) \quad (t \rightarrow +\infty), \quad (5.48)$$

$$T(e^{itu}g) \sim -i|t\lambda/\varepsilon|^{i\phi(\lambda)} e^{it\lambda} (\rho(\lambda)/\rho_1(\lambda)) \overline{c_{1,1}(k, \lambda)} g(\lambda) \quad (t \rightarrow -\infty). \quad (5.49)$$

Step 4. According to (5.48) and (5.49) we have

$$\lim_{t \rightarrow +\infty} (W_0(t) e^{it\mathcal{L}_0} e^{-it\mathcal{L}}) P = iU_0 (\rho(\lambda)/\rho_1(\lambda)) c_{1,1}(k, \lambda) U^{-1} P, \quad (5.50)$$

$$\lim_{t \rightarrow -\infty} (W_0(t) e^{it\mathcal{L}_0} e^{-it\mathcal{L}}) P = -iU_0 (\rho(\lambda)/\rho_1(\lambda)) \overline{c_{1,1}(k, \lambda)} U^{-1} P, \quad (5.51)$$

$$W_0(t) = |t\lambda/\varepsilon|^{i(\operatorname{sgn} t)\phi(\lambda)}. \quad (5.52)$$

Definition 7.1 of the generalized wave operators W_{\pm} , relations (5.50)–(5.52) and the fact that \mathcal{L}_0 and \mathcal{L} are self-adjoint, imply that

$$W_+(\mathcal{L}, \mathcal{L}_0) = -iU (\rho(\lambda)/\rho_1(\lambda)) \overline{c_{1,1}(k, \lambda)} U_0^{-1} P_0, \quad (5.53)$$

$$W_-(\mathcal{L}, \mathcal{L}_0) = iU (\rho(\lambda)/\rho_1(\lambda)) c_{1,1}(k, \lambda) U_0^{-1} P_0. \quad (5.54)$$

Therefore, in view of (7.4), the generalized dynamical scattering operator has the form

$$S_{dyn}(\mathcal{L}, \mathcal{L}_0) = U_0 ((\rho(\lambda)/\rho_1(\lambda)) c_{1,1}(k, \lambda))^2 U_0^{-1}. \quad (5.55)$$

The scattering operator $S_{dyn}(\mathcal{L}, \mathcal{L}_0)$ is unitary. Hence, it follows from (5.55) that

$$|c_{1,1}(k, \lambda)| = \rho_1(\lambda)/\rho(\lambda). \quad (5.56)$$

Formulas (5.55) and (5.56) imply the following representation of the generalized scattering operator

$$S_{dyn}(\mathcal{L}, \mathcal{L}_0) = U_0 (c_{1,1}(k, \lambda) / \overline{c_{1,1}(k, \lambda)}) U_0^{-1}, \quad \lambda > m. \quad (5.57)$$

In the same way, it can be proved that

$$S_{dyn}(\mathcal{L}, \mathcal{L}_0) = -U_0 (c_{1,1}(k, \lambda) / \overline{c_{1,1}(k, \lambda)}) U_0^{-1}, \quad \lambda < -m. \quad (5.58)$$

Recall that S_{st} satisfies (3.8) and (4.34). Then, formulas (5.1), (5.2) and (5.57), (5.58) yield (5.3) (i.e., the assertion of the theorem is proved). ■

The proof of Theorem 5.1 and Definition 7.1 imply the assertion.

Corollary 5.2 *Let the conditions of Theorem 5.1 hold. Then, the deviation factor $W_0(t)$ corresponding to the operators \mathcal{L} and \mathcal{L}_0 has the form (5.52).*

Comparing equalities (4.30) and (5.52), we obtain the assertion:

Corollary 5.3 *The deviation factor $V_0(r, \lambda)$ for the stationary case and the deviation factor $W_0(t) = W_0(t, \lambda)$ for the dynamical case are connected by the following simple equality:*

$$V_0(|t\lambda/\varepsilon|, \lambda) = W_0(t, \lambda), \quad t > 0. \quad (5.59)$$

6 The classical case ($A = 0$)

In this section, we again consider the operator \mathcal{L} of the form (4.20), (4.21), where v in (4.21) is given by (1.3), but this time we set $A = 0$ in (1.3). That is, we consider the classical case. Dynamical and stationary approaches for this case were studied separately in many important publications (see, e.g., [5, 9, 10]). Here, we compare these approaches, and our ergodic-type theorem is new even for the classical case. We stress that the classical wave and scattering operators are used in this section instead of the generalized wave and scattering operators in Section 5. The result and proof are similar to Theorem 5.1 and its proof but there are some differences, and so we consider the case $A = 0$ separately.

Theorem 6.1 *Let the radial Dirac system (1.1), (1.2), where $v(r) \equiv q(r)$ and (2.20) holds, be given.*

Then (for the corresponding operators \mathcal{L}_0 and \mathcal{L} defined via (4.9), (4.11) and (4.20)–(4.22), respectively) the generalized stationary and dynamical scattering matrices are equal on L , that is, $S_{st}(\mathcal{L}, \mathcal{L}_0)f = S_{dyn}(\mathcal{L}, \mathcal{L}_0)f$ for $f \in L$, where S_{st} , S_{dyn} and L are given in (4.34), (5.1) and (5.2), respectively.

Proof. First, we consider the case (2.2) where $\lambda > m$. Recall that ε is determined in (2.1). According to [6, Ch.8, f-las 495 and 496], the following Fourier transformation equalities are valid:

$$\int_{-\infty}^{+\infty} e^{-ir\sqrt{u^2-m^2}} \frac{(\sin(\varepsilon r/2))^2}{r} dr = \frac{\pi}{2i} \begin{cases} 1, & m < u < \lambda; \\ 0, & u > \lambda. \end{cases} \quad (6.1)$$

$$\int_{-\infty}^{+\infty} e^{-ir\sqrt{u^2-m^2}} \frac{(\sin(\varepsilon r/2))^2}{|r|} dr = \frac{1}{2} \ln \left| \frac{u^2 - \lambda^2}{u^2 - m^2} \right|. \quad (6.2)$$

From (6.1) and (6.2), we derive

$$\int_0^{+\infty} e^{-ir\sqrt{u^2-m^2}} \frac{(\sin(\varepsilon r/2))^2}{r} dr = \frac{1}{4} \ln \left| \frac{u^2 - \lambda^2}{u^2 - m^2} \right| - \frac{i\pi}{4} \begin{cases} 1, & m < u < \lambda; \\ 0, & u > \lambda. \end{cases} \quad (6.3)$$

Following the scheme of the proof of Theorem 5.1, we rewrite (5.9) in the form

$$T_2(\varepsilon, u) = \rho(u) \int_0^\infty g(r, u) \frac{2(\sin(r\varepsilon/2))^2}{r} dr. \quad (6.4)$$

In view of (6.1)–(6.3) the corresponding operator R_0 is defined by the relation

$$R_0 g = \frac{d}{d\varepsilon} \int_0^\infty g(u) \Phi(\varepsilon, u) du, \quad (6.5)$$

where

$$\Phi(\varepsilon, u) = \frac{1}{2} \rho(u) \ln \left| \frac{u^2 - \lambda^2}{u^2 - m^2} \right| - \frac{i\pi}{2} \begin{cases} \rho(u), & m < u < \lambda; \\ 0, & u > \lambda. \end{cases} \quad (6.6)$$

We represent the operator R_0 in the form

$$R_0 = R_- + R_+, \quad (6.7)$$

where

$$R_- g = \frac{1}{2} \frac{d}{d\varepsilon} \left(-i\pi \int_m^\lambda g(u) \rho(u) du + \int_m^\infty g(u) \rho(u) \ln |u - \lambda| du \right), \quad (6.8)$$

$$R_+ g = \frac{1}{2} \frac{d}{d\varepsilon} \int_m^\infty g(u) \rho(u) (\ln(u + \lambda) - \ln(u^2 - m^2)) du. \quad (6.9)$$

It is easy to see that relation (5.21) is valid in the case $A = 0$ too. Taking into account (6.8) and equality $\lambda = \sqrt{\varepsilon^2 + m^2}$, we obtain

$$R_- g = \frac{\varepsilon}{2\lambda} \left(-i\pi \rho(\lambda) g(\lambda) + \oint_m^\infty \frac{g(u) \rho(u)}{\lambda - u} du \right). \quad (6.10)$$

We note that the integral \oint on the right-hand side of (6.10) is a Cauchy-type integral. Using (6.7), (6.9), (6.10) and the equality

$$\lim_{t \rightarrow \pm\infty} \frac{i}{\pi} \oint_m^\infty f(u) \frac{e^{i(\lambda-u)t}}{\lambda - u} du = \mp f(\lambda), \quad (6.11)$$

one may show that the relations

$$R\left(e^{it\sqrt{u^2-m^2}}g\right)\sim -2i\pi e^{it\varepsilon}\rho(\lambda)g(\lambda), \quad t\rightarrow -\infty; \quad (6.12)$$

and

$$R\left(e^{it\sqrt{u^2-m^2}}g(u)\right)\sim 0, \quad t\rightarrow +\infty \quad (6.13)$$

are valid. The final part of the proof of Theorem 6.1 coincides with the final part of the proof of Theorem 5.1. ■

7 Appendix

In this Appendix, we introduce the notions of the generalized dynamical wave and scattering operators [11–13] (see also [7]). Consider linear (not necessarily bounded) operators A and A_0 acting in some Hilbert space H and assume that the operator A_0 is self-adjoint. The absolutely continuous subspace of the operator A_0 (i.e., the subspace corresponding to the absolutely continuous spectrum) is denoted by G_0 , and P_0 is the orthogonal projection on G_0 . Generalized wave operators $W_+(A, A_0)$ and $W_-(A, A_0)$ are introduced by the equality

$$W_{\pm}(A, A_0) = \lim_{t\rightarrow\pm\infty} (e^{iAt}e^{-iA_0t}W_0(t)^{-1})P_0, \quad (7.1)$$

where W_0 is an operator function taking operator values $W_0(t)$ acting in G_0 in the domain $|t| > R$ ($t \in \mathbb{R}$) for some $R \geq 0$. More precisely, we have the following definition (see [11, 12]) of the generalized wave operators $W_{\pm}(A, A_0)$ and deviation factor W_0 .

Definition 7.1 *An operator function $W_0(t)$ is called a deviation factor and operators $W_{\pm}(A, A_0)$ are called generalized wave operators if the following conditions are fulfilled:*

1. *The operators $W_0(t)$ and $W_0(t)^{-1}$ acting in G_0 , are bounded for all t ($|t| > R$), and*

$$\lim_{t\rightarrow\pm\infty} W_0(t+\tau)W_0(t)^{-1}P_0 = P_0, \quad \tau = \bar{\tau}. \quad (7.2)$$

2. *The following commutation relations hold for arbitrary values t and τ :*

$$W_0(t)A_0P_0 = A_0W_0(t)P_0, \quad W_0(t)W_0(t+\tau)P_0 = W_0(t+\tau)W_0(t)P_0. \quad (7.3)$$

3. The limits $W_{\pm}(A, A_0)$ in (7.1) exist in the sense of strong convergence.

If $W_0(t) \equiv I$ in G_0 (where I is the identity operator), then the operators $W_{\pm}(A, A_0)$ are usual wave operators.

Clearly, the choice of the deviation factor is not unique.

Remark 7.2 *Let unitary operators C_- and C_+ satisfy commutation conditions $A_0 C_{\pm} = C_{\pm} A_0$. If $W_0(t)$ is a deviation factor, then the operator function given (for $t > 0$ and $t < 0$, respectively) by the equalities $W_+(t) = C_+ W_0(t)$ ($t > 0$), and $W_-(t) = C_- W_0(t)$ ($t < 0$) is the deviation factor as well.*

The choice of the operators C_{\pm} is very important and is determined by specific physical problems. The definition below shows that generalized scattering operators also depend on the choice of C_{\pm} .

Definition 7.3 *The generalized scattering operator $S(A, A_0)$ has the form*

$$S(A, A_0) = W_+(A, A_0)^* W_-(A, A_0), \quad (7.4)$$

where

$$W_{\pm}(A, A_0) = \lim_{t \rightarrow \pm\infty} \left(e^{iAt} e^{-iA_0 t} W_{\pm}(t)^{-1} \right) P_0.$$

In fact, operator functions $W_{\pm}(t)$ are uniquely determined up to some factors $C_{\pm}(t)$ tending to C_{\pm} when t tends to ∞ or $-\infty$, respectively. This means that $S(A, A_0)$ is uniquely determined by the choice of C_{\pm} .

It is not difficult to prove that the operator $S(A, A_0)$ unitarily maps G_0 onto itself and that

$$A_0 S(A, A_0) P_0 = S(A, A_0) A_0 P_0. \quad (7.5)$$

Acknowledgements. The author is grateful to A. Sakhnovich and I. Roitberg for fruitful discussions and help in the preparation of the manuscript.

References

- [1] N.I. Akhiezer, I.M. Glazman, *Theorie der linearen Operatoren im Hilbert-Raum*, Akademie-Verlag, Berlin, 1958.

- [2] H. Bateman, A. Erdelyi, *Higher Transcendental Functions*, I, McGraw–Hill Book Company Inc., New York, 1953.
- [3] V.B. Berestetskii, E.M. Lifshits, L.P. Pitaevskii, *Quantum electrodynamics*, Pergamon Press, Oxford, 1982.
- [4] R. Bellman, *Stability theory of differential equations*, McGraw–Hill Book Company, New York, 1953.
- [5] M.Sh. Birman, M.G. Krein, *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR **144**:3 (1962), 475–478.
- [6] Yu.A. Brychkov, A.P. Prudnikov, *Integral transforms of generalized functions* (Russian), Nauka, Moscow, 1977.
- [7] V.S. Buslaev, V.B. Matveev, *Wave operators for the Shrödinger equation with a slowly decreasing potential*, Theor. Math. Fiz. **2**:3 (1970), 367–376.
- [8] I.M. Gelfand, G.E. Shilov, *Verallgemeinerte funktionen (Distributionen)*, Deutscher Verlag der Wissenschaften, Berlin, 1960.
- [9] T. Kato, *Wave operators and unitary equivalence*, Pacific J. Math. **15** (1965), 171–180.
- [10] M. Rosenblum, *Perturbation of the continuous spectrum and unitary equivalence*, Pacific J. Math. **7** (1957), 997–1010.
- [11] L.A. Sakhnovich, *Dissipative operators with absolutely continuous spectrum*, Trans. Moscow Math. Soc. **19** (1968), 233–297.
- [12] L.A. Sakhnovich, *Generalized wave operators*, Math. USSR Sbornik **10**:2 (1970), 197–216.
- [13] L.A. Sakhnovich, *The invariance principle for generalized wave operators*, Funct. Anal. Appl. **5**:1 (1971), 49–55.
- [14] L.A. Sakhnovich, *Spectral theory of canonical differential systems, method of operator identities*, Operator Theory Adv. Appl. **107**, Birkhäuser, Basel, 1999.